Lecture 1

1 Introduction

Probability is a branch of Mathematics which deals with random experiments and we will start our journey in probability with what is known as sample space (represented mathematically by Ω) which describes all possible outcomes in the random experiment.

Examples of random experiments and their sample spaces:

1. Tossing a fair coin once
$$\Omega = \{H, T\},$$
where H represents head and T represents tail.

2. Tossing a fair coin twice
$$\Omega = \{HH, HT, TH, TT\}.$$

3. Rolling a fair dice once
$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

4. Number of emails received at your inbox in the upcoming year
$$\Omega = \{0, 1, 2, 3, \cdots\}.$$

5. Waiting trials until you get number 6 when you throw a fair dice
$$\Omega = \{0, 1, 2, 3, \cdots\},$$
where 0 means number 6 appears immediately in the first throw, 1 means you have missed the first toss and number 6 appears in the second trial and 3 means you have missed the first two tosses and number 6 appears in the third trial and so on.

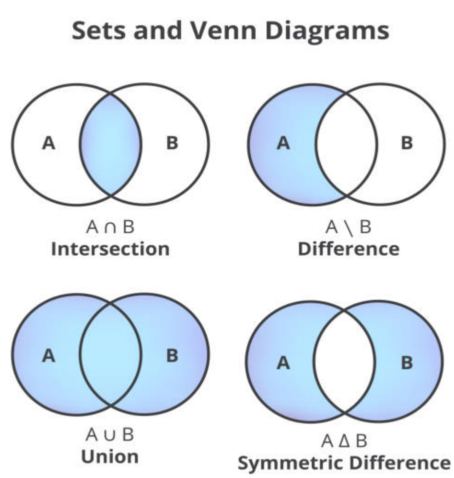
6. Waiting time for the bus at bus station
$$\Omega = [0, \infty),$$
where ∞ means that you have waited the bus for a long time

Therefore, we can conclude from the above examples that sample space Ω can take discrete values as in third, fourth and fifth examples or continuous values as in sixth example and it can be also finite as in first, second and third examples or infinite as in fourth, fifth and sixth examples.

Mathematically speaking, sample space is a set. If Ω is discrete and finite sample space and A is a subset of Ω ($A \subseteq \Omega$), then we call A an event. However, we have to take care in case of infinite or continuous sample spaces because not not every subsets of these sample spaces are called events and there are other issues and mathematical technicalities that need to be taken into consideration first before considering them events or not. This is a separate story outside the scope of our discussion. Here, we will deal with discrete and finite sample spaces. Therefore any subsets of these sample spaces are considered events.

If A and B are two events, then

- $A \cup B = \{x \in \Omega : x \in A \quad \text{or} \quad x \in B\}$, where x are elements of the sample space that exist in A or B.
- $A \cap B = \{x \in \Omega : x \in A \quad \text{and} \quad x \in B\}$, where x are elements of the sample space that exist in A and B.
- $A \setminus B = \{x \in \Omega : x \in A \quad \text{and} \quad x \notin B\}$, where x are elements of the sample space that exist in A and not in B.
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$, where $A \triangle B$ is called symmetric difference of A and B which means elements of the sample space that exist in A and not in B or elements of the sample space that exist in B and not in A.



2 Axioms of Probability:

1. In probability, we assign values to events. In other words, probability is an assignment of a value to each event in the sample space. This value must lie in interval $[0, 1]$ and $\mathbb{P}(A)$ denotes the value assigned to the event A

2. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\Phi) = 0$, where Φ is the empty set which is always a subset of the sample space Ω.

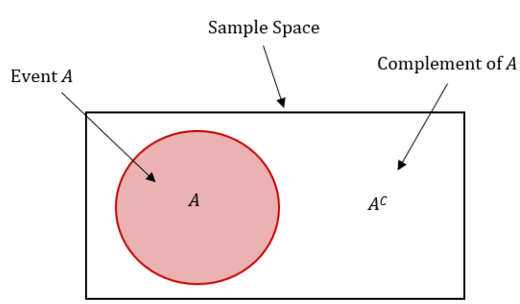
3. If $A_1, A_2, \cdots, A_n$ are mutually disjoints events which means that they do not have any elements in common between them and they do not overlap and their intersection is the empty set Φ. Then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i),$$

Example: Let A be an event and A^c denotes the complement of the event A which means the remaining part of the the whole sample space Ω as shown in the figure Then, we have

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Proof: Since A and A^c are mutually disjoints events $(A \cap A^c = \Phi)$ and form



the whole sample space Ω $(A \cup A^c = \Omega)$. Then, we can utilize the second and third axioms of probability as follows

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cap A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1.$$

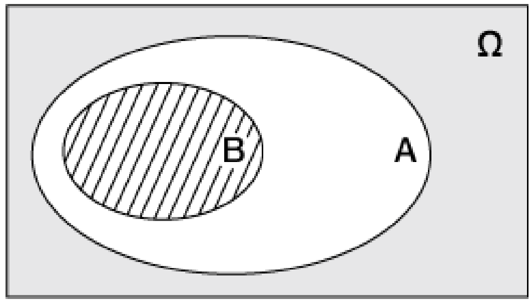
Therefore,

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

Example: Let A and B be two events in the sample space Ω such that $B \subset A \subset \Omega$. Then

$$\mathbb{P}(B) \leq \mathbb{P}(A)$$

Proof: Since the events B and $(A \setminus B)$ (white area) are mutually disjoints



$$B \cap (A \setminus B) = \Phi,$$

and their union form the whole event A as follows

$$A = B \cup (A \setminus B).$$

Therefore,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(B \cup (A \setminus B)) \\ &= \mathbb{P}(B) + \mathbb{P}(A \setminus B).\end{aligned}$$

Since $0 \leq \mathbb{P}(A \setminus B) \leq 1$, we can conclude that

$$\mathbb{P}(B) \leq \mathbb{P}(A).$$

3 Counting

Counting gives us a way to compute probabilities of events when the probability of each outcome in the sample space is the same (they are equally likely) and there is no any prior information regarding any one of the outcomes. Therefore for a sample space having all outcomes of same probability or equally likely, we have the following:

- The sample space Ω cannot be infinite.
- The probability of each outcome in the sample space Ω is $1/n$, where n is the total number of the outcomes in the sample space.
- If $A \subset \Omega$, then $\mathbb{P}(A) = k/n$, where k is the total number of outcomes in event A and n is the total number of the outcomes in the sample space Ω. Therefore we need techniques for counting to compute probabilities.

Multiplication Principle: If the random experiment 1 has m possible outcomes and the random experiment 2 has n possible outcomes. Then experiments 1 and 2 have $m * n$ possible outcomes.

Example: If I toss a coin then roll a fair dice. How many possible outcomes are there?
Ans. It will be 2*6=12 possible outcomes.

Example: Find the number of ways to form an ordered sample without replacement from a group of 47 people to do different 3-tasks?
Ans. It will be 47*46*45 options.

Example: Find the number of ways to form an ordered sample with replacement from a group of 47 people to do different 3-tasks?
Ans. It will be 47*47*47 options.

Example: How many ways can a committe of size 5 be selected from a group of 47 people without replacement?
Ans. It will be $(47 * 46 * 45 * 44 * 43)/5! = \binom{47}{5}$ options.

Example: How many ways are there to select r objects from a group of n objects without replacement where the order does not matter?
Ans. It will be

$$\frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Example: How many ways are there to select r objects from a group of n objects without replacement where the order does matter?
Ans. It will be

$$n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!} = {}^{n}P_{r}.$$

Example: In the 52-playing cards, find the probability of getting Full house (for example 33355 or 77788) and the probability of getting Flush (all cards of the same suit)?
Ans. It will be

$$\mathbb{P}(\text{Full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}},$$

and

$$\mathbb{P}(\text{Flush}) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}.$$